Angular Synchronization and its application in Phase Retrieval

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joint work with
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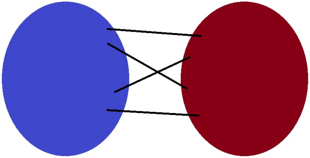
OSL 2013, Les Houches.
January 11, 2013

http//www.math.princeton.edu/~ajsb

$$G = \big(V, E, (W)_{ij} = w_{ij}\big)$$



Cheeger Constant:

$$h_G = \min_{S \subset V} h_G(S)$$

$$h_G(S) = \frac{\mathrm{cut}(S, S^c)}{\min\{\mathrm{vol}(S), \mathrm{vol}(S^c)\}}$$

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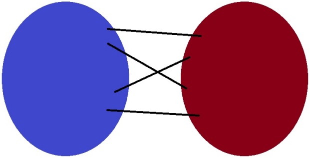

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Graph Laplacian

$$D = diag(d_i)$$

$$L_0 = D - W \quad \text{and} \quad \mathcal{L}_0 = I - D^{-1/2} W D^{-1/2}$$

$$\frac{x^T L_0 x}{x^T D x} = \frac{1}{2} \frac{\sum_{ij} w_{ij} |x_i - x_j|^2}{\sum_i d_i x_i^2}$$

Spectral Clustering – Cheeger Inequality

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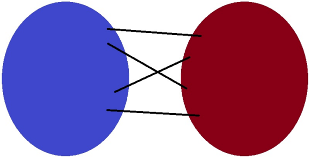

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Theorem (Cheeger Inequality (Alon 86))

$$\frac{1}{2}\lambda_2(\mathcal{L}_0) \le h_G \le \sqrt{2\lambda_2(\mathcal{L}_0)}$$

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f a function that takes values in $[0, 1]$.

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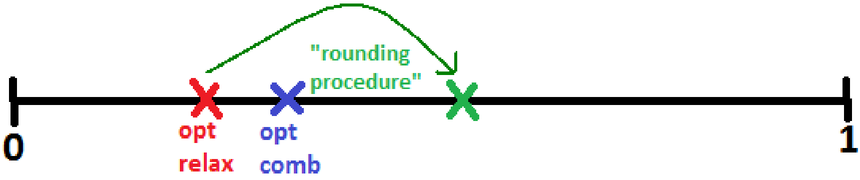

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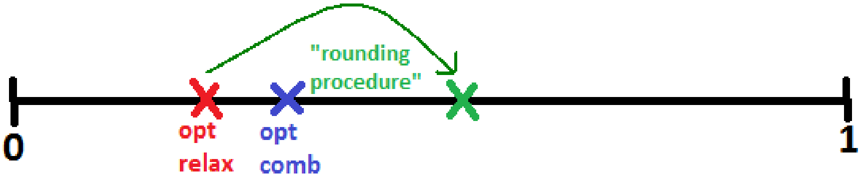

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$$\text{opt relax} \leq \text{opt comb} \leq 2\,(\text{ opt relax })$$

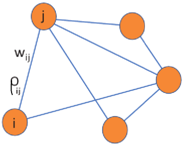
Problem

Determine a potential on the set V of vertices of a graph, with values on a group $\mathcal{G}$

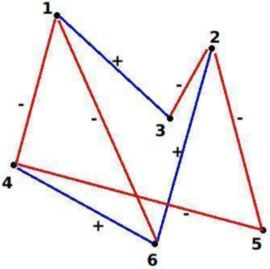
$$\begin{aligned} g : V &\to \mathcal{G} \\ i &\to g_i \end{aligned}$$

given a few, possibly noisy, of the pairwise offset measurements (corresponding to the edges E of the graph)

$$\begin{aligned} \rho : E &\to \mathcal{G} \\ (i,j) &\to \rho_{ij} \approx g_i g_j^{-1}. \end{aligned}$$



Examples... $\mathcal{G} = O(1) = \mathbb{Z}_2$



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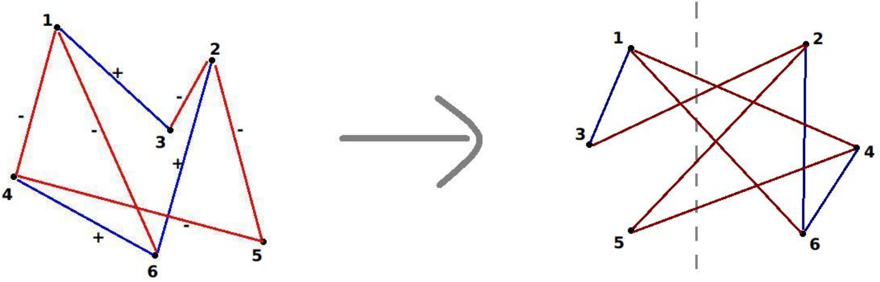

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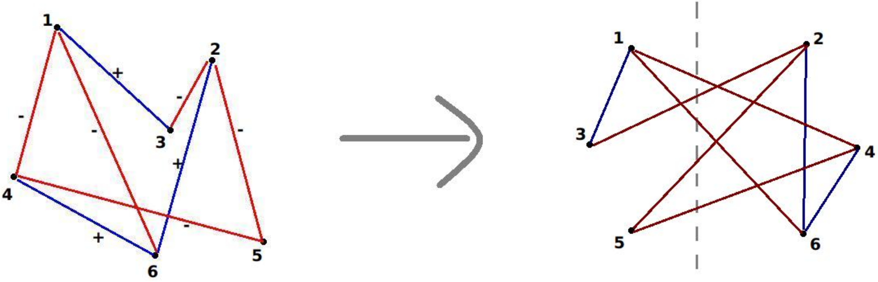
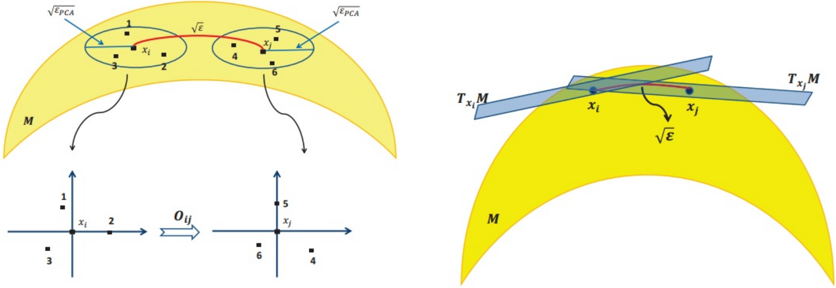


When all edges are red this is essentially Max-Cut

Orientation of a Manifold.



$$\rho_{ij} = \det(O_{ij})$$

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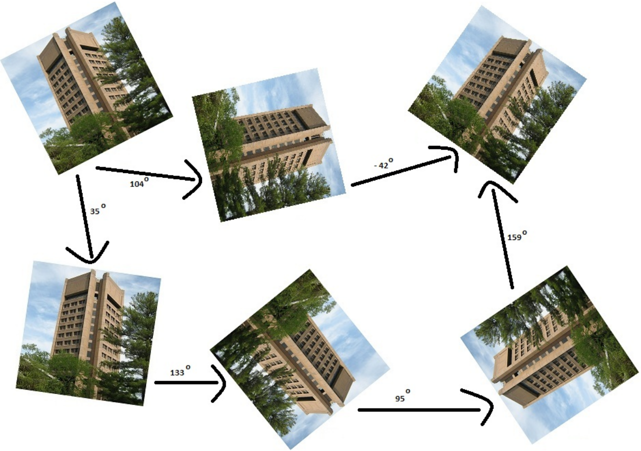

Examples... $\mathcal{G} = SO(2)$

333

77

35

8

141

236

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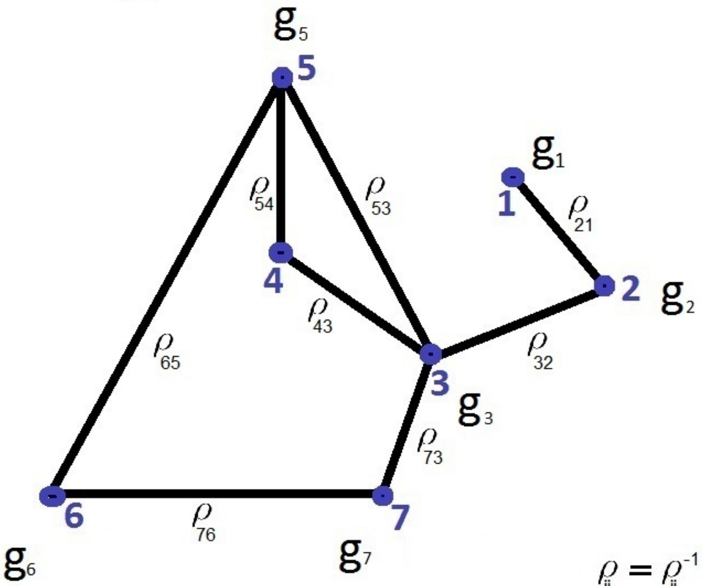
Solution to the "frustration free" case



$\rho_{ij} = \rho_{ji}^{-1}$

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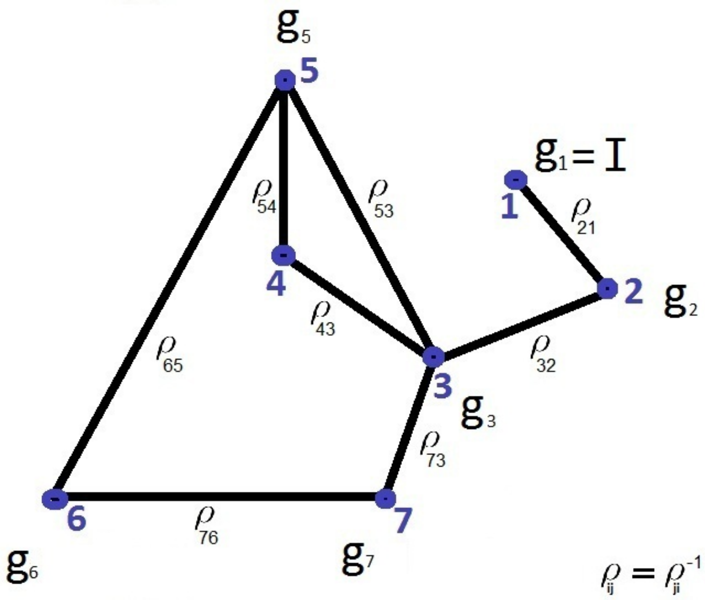

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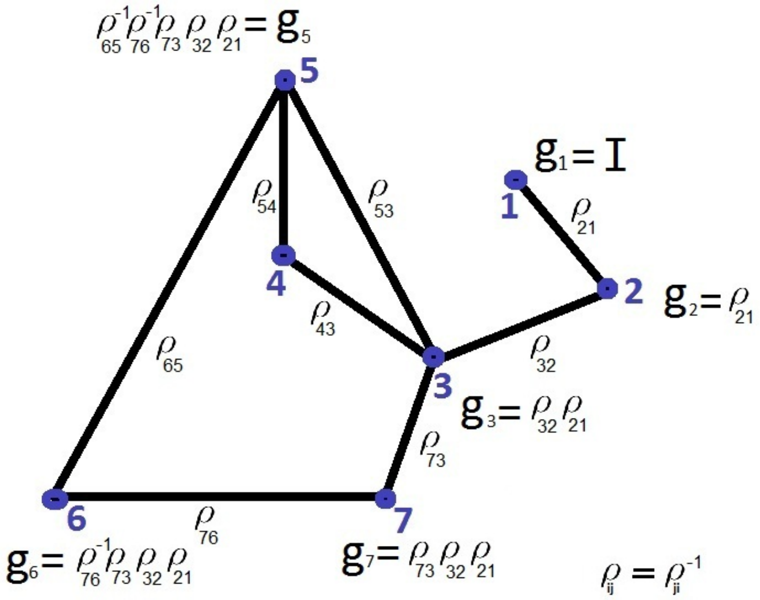

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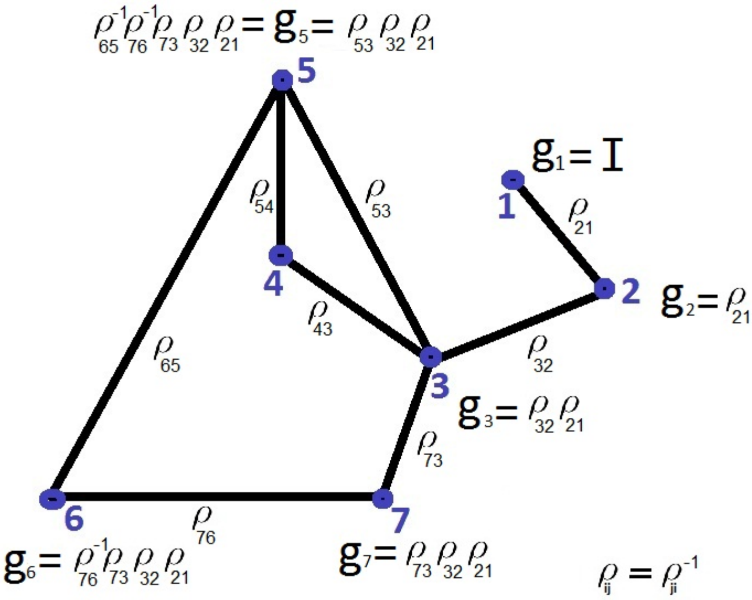

The Angular Synchronization Problem

Problem

Determine an angular potential on the set V of vertices of a graph,

$$\begin{aligned} \theta_\cdot : V &\to [0, 2\pi) \\ i &\to \theta_i \end{aligned} \qquad\qquad \begin{aligned} e^{i\theta_\cdot} = v : V &\to \mathbb{T} \subset \mathbb{C} \\ i &\to v_i \end{aligned}$$

given a few, possibly noisy, of the relative angle measurements (corresponding to the edges E of the graph)

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The Frustration Constant:

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The Graph Connection Laplacian

$W_1 \in \mathbb{C}^{n \times n}$

$$(W_1)_{ij} = w_{ij}\rho_{ij} \in \mathbb{C}.$$

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The Normalized Graph Connection Laplacian is $\mathcal{L}_1 \in \mathbb{C}^{n\times n}$

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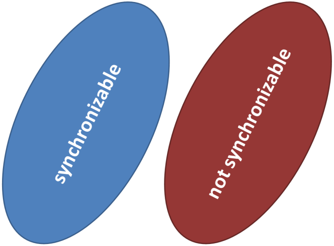

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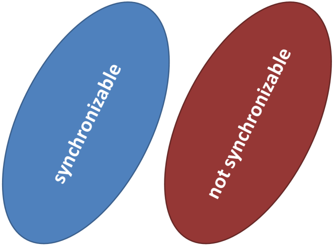

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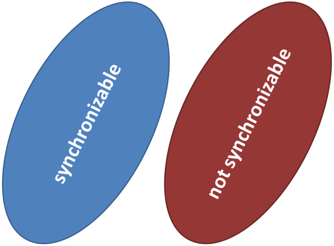

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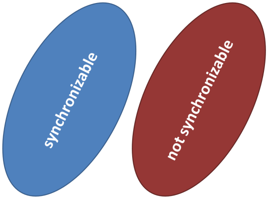
Theorem

$$\lambda_1(\mathcal{L}_1) \leq \eta_G^* \leq \sqrt{10 \lambda_1(\mathcal{L}_1)}$$

Global Synchronization – What about η_G?

Problematic case:



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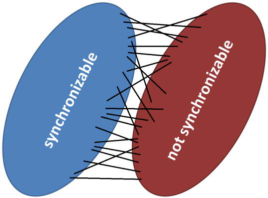

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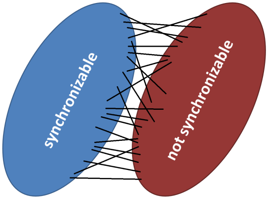

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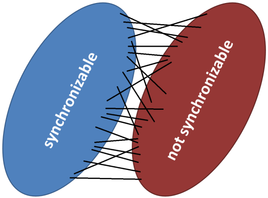

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$$\geq \frac{1}{2}\frac{\sum_{ij} w_{ij}\left(|v_i| - |v_j|\right)^2}{\sum_i d_i|v_i|^2}$$

Theorem

$$\lambda_1(\mathcal{L}_1) \leq \eta_G \leq \frac{1}{\lambda_2(\mathcal{L}_0)}\mathcal{O}\Big(\lambda_1(\mathcal{L}_1)\Big).$$

Examples... $\mathcal{G} = SO(3)$

What about beyond $\mathbb{Z}/2\mathbb{Z} = O(1)$ and $SO(2)$ Synchronization?

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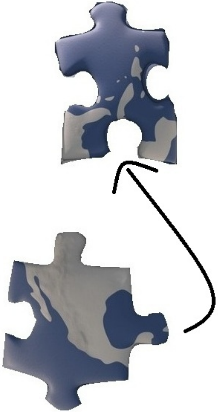

Higher-Order Rotation Groups

We want to globally estimate $O : V \to O(d)$ such that $O_i \approx \rho_{ij} O_j$.
Minimize:

$$\nu(O) = \frac{1}{\mathrm{vol}(G)} \sum_{ij} w_{ij} \|O_i - \rho_{ij} O_j\|_F^2.$$

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Many eigenvalues/eigenvectors are needed

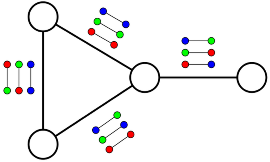
Theorem

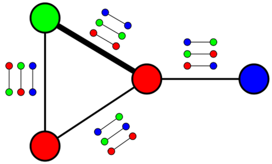
Let $\lambda_i(\mathcal{L}_1)$ and $\lambda_i(\mathcal{L}_0)$ denote the i-th smallest eigenvalue of, respectively, the normalized Connection Laplacian $\mathcal{L}_1$ and the normalized graph Laplacian $\mathcal{L}_0$. Let ν_G denote the $O(d)$ frustration constant of G. Then,

$$\frac{1}{d} \sum_{i=1}^{d} \lambda_i(\mathcal{L}_1) \le \nu_G \le \mathrm{poly}(d) \frac{1}{\lambda_2(\mathcal{L}_0)} \sum_{i=1}^{d} \lambda_i(\mathcal{L}_1).$$

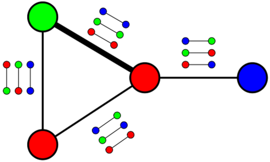
The proof is constructive – the Algorithm achieves this!

The Unique Games Conjecture





Let opt be the minimum fraction of edges the coloring gets wrong.



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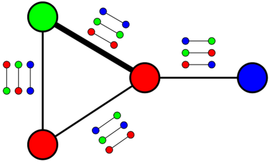
Conjecture (U.G.C.)

For every $\epsilon \sim 0$ and $\delta \sim 1$ there exists k and an assignement of the edges (with k colors) such that deciding whether

$$\mathrm{opt} < \epsilon \quad \textit{or} \quad \mathrm{opt} > \delta$$

is **NP**–*hard.*



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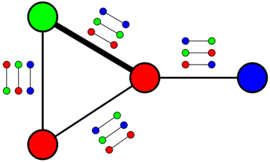
- Corresponds to localization in S_k.

  One can represent S_k as permutation matrices in $O(k)$.



Let opt be the minimum fraction of edges the coloring gets wrong.

Conjecture (U.G.C.)

For every $\epsilon \sim 0$ and $\delta \sim 1$ there exists k and an assignement of the edges (with k colors) such that deciding whether

$$\mathrm{opt} < \epsilon \quad \textit{or} \quad \mathrm{opt} > \delta$$

is **NP**–*hard.*

- Corresponds to localization in S_k.

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- There seems to be NO good "rounding procedure".

  e.g.: all-ones vector is a perfect localization for relaxed problem

PART II:

Reconstruction without phase

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- A signal $x \in \mathbb{C}^M$ is measured using a linear system but only the absolute value of the measurements is obtained

$$|\langle x, \varphi_n \rangle|, \quad n = 1, \ldots, N$$

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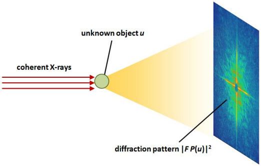

Motivation: X-ray Crystallography and inversion of spectrograms.

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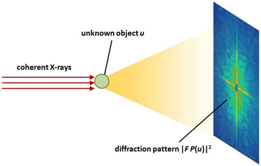

Motivation: X-ray Crystallography and inversion of spectrograms.

Question

When and how can we reconstruct x from these phaseless measurements?

State of the art

- Balan et al., 2006: For a generic system, phaseless measurements are injective whenever $N \geq 4M - 2$
  . The right injectivity bound is believed to be $4M - 4$

- Phaselift (Candès et al., 2011) and Phasecut (Waldspurger et al., 2012): For a random system, stable recovery by Semi-Definite Programming for $N = \tilde{O}(M)$.

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Question

Can we design a measurement matrix such that it is possible to efficiently and stably recovery from only $N = \tilde{O}(M)$ measurements avoiding the SDP computational cost?

Polarization

- Synchronization allows to recover the phases of the measurements from the relative phases

$$\omega_{ij} := \Big(\frac{\langle x, \varphi_i\rangle}{|\langle x, \varphi_i\rangle|}\Big)^{-1}\frac{\langle x, \varphi_j\rangle}{|\langle x, \varphi_j\rangle|} = \frac{\overline{\langle x, \varphi_i\rangle}\langle x, \varphi_j\rangle}{|\langle x, \varphi_i\rangle||\langle x, \varphi_j\rangle|}$$

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- We can determine ω_{ij} from other phaseless measurements:

$$\overline{\langle x, \varphi_i\rangle}\langle x, \varphi_j\rangle = \frac{1}{4}\sum_{k=1}^{4} \mathrm{i}^k |\langle x, \varphi_i + \mathrm{i}^k \varphi_j\rangle|^2$$

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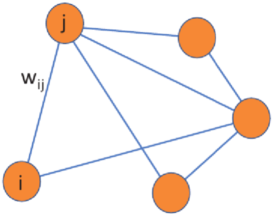

- each φ_i corresponds to vertex i
- each set $\{\varphi_i + \mathrm{i}^k \varphi_j\}_{k=1}^4$ to an edge between i and j.

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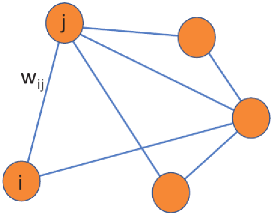

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- We need a sparse graph!

Instability of near orthogonality

- The measurements are noisy $\left(|\langle x, \varphi_i\rangle| + \epsilon_i\right)$.
- If x is nearly orthogonal to φ_i the noise in the relative phase blows-up

$$\omega_{ij} = \frac{\overline{\langle x, \varphi_i\rangle}\langle x, \varphi_j\rangle + \epsilon_{ij}}{\left|\overline{\langle x, \varphi_i\rangle}\langle x, \varphi_j\rangle + \epsilon_{ij}\right|}$$

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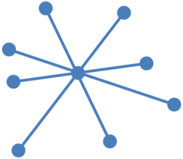
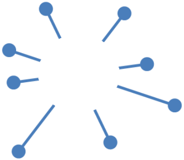
  1. **FRAME DESIGN** - do not delete too many vertices
  2. **GRAPH DESIGN** - still be able to perform synchronization





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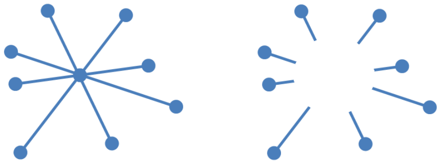


Solution:

1. Gaussian Measurements
2. Expander graphs

Stability of Phaseless reconstruction

Theorem (Alexeev-Bandeira-Fickus-M, 2012)

Take $N \sim CM \log M$ with C sufficiently large. Then the following holds for all $x \in \mathbb{C}^M$ with overwhelming probability:

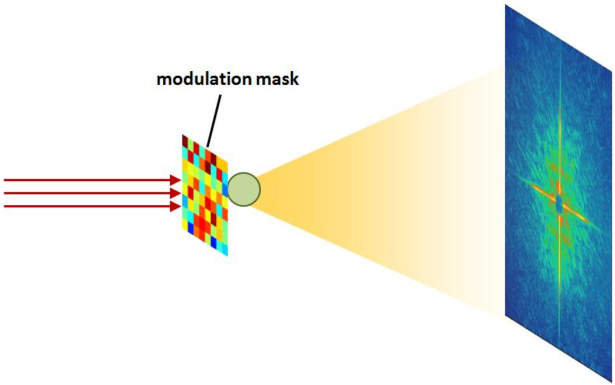
Given noisy intensity measurements

$$z_\ell := |\langle x, \varphi_\ell \rangle|^2 + \nu_\ell,$$

if the noise-to-signal ratio satisfies $\mathrm{SNR} := \frac{\|x\|_2^2}{\|\nu\|_2} \geq \frac{\sqrt{M}}{C'}$, then our phase retrieval procedure produces $\tilde{x}$ with squared relative error

$$\frac{\|\tilde{x} - \mathrm{e}^{\mathrm{i}\theta} x\|_2^2}{\|x\|_2^2} \leq K \sqrt{\frac{M}{\log M}}\ \mathrm{SNR}^{-1},$$

for some phase $\theta \in [0, 2\pi)$.



We were able to design $\mathcal{O}(\log M)$ Fourier Masks providing measurements that allow for reconstruction with the polarization algorithm,

both the vertex and edge measurements are contained in those $\mathcal{O}(\log M)$ designed Fourier Masks.

Thank You

A. S. Bandeira, A. Singer and D. A. Spielman,
"A Cheeger Inequality for the Graph Connection Laplacian"
arXiv:1204.3873

B. Alexeev, A. S. Bandeira, D. G. Mixon, and M. Fickus,
"Phase retrieval with polarization"
arXiv:1210.7752