Critical Motions for Auto-Calibration When Some Intrinsic Parameters Can Vary *

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Abstract.
Auto-calibration is the recovery of the full camera geometry and Euclidean scene structure from several images of an unknown 3D scene, using rigidity constraints and partial knowledge of the camera intrinsic parameters. It fails for certain special classes of camera motion. This paper derives necessary and sufficient conditions for unique auto-calibration, for several practically important cases where some of the intrinsic parameters are known (e.g. skew, aspect ratio) and others can vary (e.g. focal length). We introduce a novel subgroup condition *on the camera calibration matrix, which helps to systematize this sort of auto-calibration problem. We show that for subgroup constraints, criticality is independent of the exact values of the intrinsic parameters and depends only on the camera motion. We study such* critical motions *for arbitrary numbers of images under the following constraints: vanishing skew, known aspect ratio and full internal calibration modulo unknown focal lengths. We give explicit, geometric descriptions for most of the singular cases. For example, in the case of unknown focal lengths, the only critical motions are: (i) arbitrary rotations about the optical axis and translations, (ii) arbitrary rotations about at most two centres, (iii) forward-looking motions along an ellipse and/or a corresponding hyperbola in an orthogonal plane. Some practically important special cases are also analyzed in more detail.*

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1. Introduction

One of the core problems in computer vision is the recovery of 3D scene structure and camera motion from a set of images. However, for certain configurations there are inherent ambiguities. This kind of problem was already studied in optics in the early 19th century, for example, by Vieth in 1818 and Muller in 1826. Pioneering work on the subject was also done by Helmholtz. See [13] for references. One well-studied ambiguity is when the visible features lie on a special surface, called a **critical surface**, and the cameras have a certain position relative to the surface. Critical surfaces or "gefährlicher Ort" were studied by Krames [21] based on a monograph from 1880 on quadrics [32]. See also the book by Maybank [24] for a more recent treatment. Another well-known ambiguity is that when using projective image measurements, it is only possible to recover the scene up to an unknown projective transformation [8, 10, 35]. Additional scene, motion or calibration constraints are required for a (scaled) Euclidean reconstruction. **Auto-calibration** uses qualitative constraints on the camera calibration, e.g. vanishing skew or unit aspect ratio, to reduce the projective ambiguity to a similarity. Unfortunately, there are situations when the auto-calibration constraints may lead to several possible Euclidean reconstructions. In this paper, such degeneracies are studied under various auto-calibration constraints.

In general it is possible to recover Euclidean scene information from $m \geq 3$ images by assuming constant but unknown intrinsic parameters of a moving projective camera [26, 7]. Several practical algorithms have been developed [39, 2, 30]. Some of the intrinsic parameters may even vary, e.g. the focal length [31], or the focal length and the principal point [14]. In [29, 15] it was shown that vanishing skew suffices for a Euclidean reconstruction. Finally in, [16] it was shown that given at least 8 images it is sufficient if just one of the intrinsic parameters is known to be constant (but otherwise unknown).

However, for certain camera motions, these auto-calibration constraints are *not* sufficient [42, 1, 40]. A complete categorization of these **critical motions** in the case of constant intrinsic parameters was given by Sturm [36, 37]. The uniformity of the constant-intrinsic constraints makes this case relatively simple to analyze. But it is also somewhat unrealistic: It is often reasonable to assume that the skew actually van-

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ishes whereas focal length often varies between images. While the case of constant parameters is practically solved, much less is known for other auto-calibration constraints. In [43], additional scene and calibration constraints are used to resolve ambiguous reconstructions, caused by a fixed axis rotation. The case of two cameras with unknown focal lengths is studied in [12, 28, 4, 20]. For the general unknown focal length case, Sturm [38] has independently derived results similar to those presented here and in [20, 19].

In this paper, we generalise the work of Sturm [37] by relaxing the constraint constancy on the intrinsic parameters. We show that for a large class of auto-calibration constraints, the degeneracies are independent of the specific values of the intrinsic parameters. Therefore, it makes sense to speak of *critical motions* rather than critical configurations. We then derive the critical motions for various auto-calibration constraints. The problem is formulated in terms of projective geometry and the absolute conic. We start with fully calibrated cameras, and then continue with cameras with unknown and possibly varying focal length, principal point, and finally aspect ratio. Once the general description of the degenerate motions has been completed, some particular motions frequently occurring in practice are examined in more detail.

This paper is organized as follows. In Section 2 some background on projective geometry for vision is presented. Section 3 gives a formal problem statement and reformulates the problem in terms of the absolute conic. In Section 4, our general approach to solving the problem is presented, and Section 5 derives the actual critical motions under various auto-calibration constraints. Some particular motions are analyzed in Section 6. In order to give some practical insight of critical and near-critical motions, some experiments are presented in Section 7. Finally, Section 8 concludes.

2. Background

In this section, we give a brief summary of the modern projective formulation of visual geometry. Also, some basic concepts in projective and algebraic geometry are introduced. For further reading, see [6, 24, 33].

A **perspective (pinhole) camera** is modeled in homogeneous coordinates by the projection equation

$$\boldsymbol{x} \simeq \boldsymbol{P}\mathbf{X} \qquad (1)$$

where $\mathbf{X} = (X, Y, Z, W)^T$ is a 3D world point, $\boldsymbol{x} = (x, y, z)^T$ is its 2D image, $\boldsymbol{P}$ is the 3×4 camera **projection matrix** and $\simeq$ denotes equality up to scale. Homogeneous coordinates are used for both image and object coordinates. In a Euclidean frame, $\boldsymbol{P}$ can be factored, using a QR-decomposition, cf. [9], as

$$\boldsymbol{P} \simeq \boldsymbol{K}\,[\boldsymbol{R}\,|\,-\boldsymbol{R}\mathbf{t}] \quad \text{where} \quad \boldsymbol{K} = \begin{bmatrix} f & f\,s & u_0 \\ 0 & f\,\gamma & v_0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Here the **extrinsic parameters** $(\boldsymbol{R}, \mathbf{t})$ denote a 3×3 rotation matrix and a 3×1 translation vector, which encode the pose of the camera. The columns of $\boldsymbol{R} = [\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3]$ define an orthogonal base. The standard base is defined by $\mathbf{e}_1 = (1,0,0)^T, \mathbf{e}_2 = (0,1,0)^T, \mathbf{e}_3 = (0,0,1)^T$. The **intrinsic parameters** in the **calibration matrix** $\boldsymbol{K}$ encode the camera's internal geometry: f denotes the **focal length**, γ the **aspect ratio**, s the **skew** and (u_0, v_0) the **principal point**.

A camera for which $\boldsymbol{K}$ is unknown is said to be **uncalibrated**. It is well-known that for uncalibrated cameras, it is only possible to recover the $3D$ scene and the camera poses up to unknown projective transformation [8, 10, 35]. This follows directly from the projection equation (1): Given one set of camera matrices and $3D$ points that satisfies (1), another reconstruction can be obtained from

$$\boldsymbol{P}\,\mathbf{X} = (\boldsymbol{P}\,T)\,(T^{-1}\,\mathbf{X}) = \tilde{\boldsymbol{P}}\,\tilde{\mathbf{X}},$$

where T is a non-singular 4×4 matrix corresponding to a projective transformation of $\mathbb{P}^3$.

A quadric in $\mathbb{P}^n$ is defined by the quadratic form

$$\boldsymbol{X}^T \boldsymbol{Q} \boldsymbol{X} = 0,$$

where $\boldsymbol{Q}$ denotes a $(n+1) \times (n+1)$ symmetric matrix and $\boldsymbol{X}$ denotes homogeneous point coordinates. The **dual** is a quadric envelope, given by

$$\Pi^T \boldsymbol{Q}^* \Pi = 0, \tag{3}$$

where Π denotes homogeneous coordinates for hyper-planes of dimension $n-1$ that are tangent to the quadric. For non-singular matrices, it can be shown that $\boldsymbol{Q} \simeq (\boldsymbol{Q}^*)^{-1}$ (see [33] for a proof). A quadric with a non-singular matrix is said to be **proper**. Quadrics with no real points are called **virtual**. In the plane, $n = 2$, quadrics are called **conics**. We will use $\boldsymbol{C}$ for the 3×3 matrix that defines the conic points $\boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} = 0$ and $\boldsymbol{C}^*$ for its dual that defines the envelope of tangent lines $\boldsymbol{l}^T \boldsymbol{C}^* \boldsymbol{l} = 0$ (where $\boldsymbol{C}^* \simeq \boldsymbol{C}^{-1}$). The image of a quadric in 3D-space is a conic, i.e. the silhouette of a 3D quadric is projected to a conic curve. This can be expressed in envelope forms as

$$\boldsymbol{C}^* \simeq \boldsymbol{P} \boldsymbol{Q}^* \boldsymbol{P}^T. \tag{4}$$

Projective geometry encodes only cross ratios and incidences. Properties like parallelism and angles are not invariant under different projective coordinate systems. An affine space, where properties like parallelism and ratios of lengths are preserved, can be embedded in a projective space by singling out a **plane at infinity** Π_∞. The points on Π_∞ are called points at infinity and be interpreted as direction vectors. In $\mathbb{P}^3$, Euclidean properties, like angles and lengths, are encoded by singling out a proper, virtual conic on Π_∞. This **absolute conic** Ω_∞ gives scalar products between direction vectors. Its dual, the **dual absolute quadric** Ω^*_∞, gives scalar products between plane normals. Ω^*_∞ is a 4×4 symmetric rank 3 positive semidefinite matrix, where the coordinate system is normally chosen such that $\Omega^*_\infty = \mathrm{diag}(1,1,1,0)$. Π_∞ is Ω^*_∞'s unique null vector: $\Omega^*_\infty \Pi_\infty = 0$. The **similarities** or scaled **Euclidean transformations** in projective space are exactly those transformations that leave Ω_∞ invariant. The transformations that leave Π_∞ invariant are the **affine transformations**. The different forms of the absolute conic will be abbreviated to (D)AC for (*D*ual) *A*bsolute *C*onic.

Given image conics in several images, there may or may not be a 3D quadric having them as image projections. The constraints which guarantee this in two images are called the **Kruppa constraints** [22]. In the two-image case, these constraints have been successfully applied in order to derive the critical sets, e.g. [28]. For the more general case of multiple images, the projection equation given by (4) can be used for each image separately.

3. Problem Formulation

The problem of auto-calibration is to find the intrinsic camera parameters $(\boldsymbol{K}_i)_{i=1}^m$, where m denotes the number of camera positions. In general, auto-calibration algorithms proceed from a projective reconstruction of the camera motion. In order to auto-calibrate, some constraints have to be enforced on the intrinsic parameters, e.g. vanishing skew and/or unit aspect ratio. Thus, we require that the calibration matrices should belong to some proper subset $\mathcal{G}$ of the group $\mathcal{K}$ of 3×3 upper triangular matrices. Once the projective reconstruction and the intrinsic parameters are known, Euclidean structure and motion are easily computed.

For a general set of scene points seen in two or more images, there is a unique projective reconstruction. However, certain special configurations, known as critical surfaces, give rise to additional ambiguous solutions. For two cameras, the critical configurations occur only if both camera centres and all scene points lie on a ruled quadric surface [24].

Furthermore, when an alternative reconstruction exists, then there will always exist a third distinct reconstruction. For more than two cameras, the situation is less clear. In [25], it is proven that when six scene points and any number of camera centres lie on a ruled quadric, then there are three distinct reconstructions. If there are other critical surfaces is an open problem.

We will avoid critical surfaces by assuming unambiguous recovery of projective scene structure and camera motion. In other words, the camera matrices and the 3D scene are considered to be known up to an unknown projective transformation. We formulate the auto-calibration problem as follows: If all that is known about the camera motions and calibrations is that each calibration matrix $\boldsymbol{K}_i$ lies in some given constraint set $\mathcal{G} \subset \mathcal{K}$, when is a unique auto-calibration possible? More formally:

Problem 3.1. *Let $\mathcal{G} \subset \mathcal{K}$. Then, given the true camera projections $(\boldsymbol{P}_i)_{i=1}^m$, where $\boldsymbol{P}_i = \boldsymbol{K}_i[\boldsymbol{R}_i \,|\, -\boldsymbol{R}_i\mathbf{t}_i\,]$ and $\boldsymbol{K}_i \in \mathcal{G}$, is there any projective transformation T (not a similarity) such that $\tilde{\boldsymbol{P}}_i \simeq \boldsymbol{P}_i T$ has decomposition $\tilde{\boldsymbol{P}}_i = \tilde{\boldsymbol{K}}_i[\tilde{\boldsymbol{R}}_i \,|\, -\tilde{\boldsymbol{R}}_i\tilde{\mathbf{t}}_i\,]$ with calibration matrices $\tilde{\boldsymbol{K}}_i$ lying in $\mathcal{G}$?*

Without constraints on the intrinsic parameters T can be chosen arbitrarily, so auto-calibration is impossible. Also, T is only defined modulo a similarity,

$$T \to T \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix},$$

as such transformations leave $\boldsymbol{K}$ in the decomposition $\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{R} \,|\, -\boldsymbol{R}\mathbf{t}\,]$ invariant. Based on the above problem formulation, we can define precisely what is meant by a motion being critical.

Definition 3.1. Let $\mathcal{G} \subset \mathcal{K}$ and let $(\boldsymbol{P}_i)_{i=1}^m$ and $(\tilde{\boldsymbol{P}}_i)_{i=1}^m$ denote two projectively related motions, with calibration matrices $(\boldsymbol{K}_i)_{i=1}^m$ and $(\tilde{\boldsymbol{K}}_i)_{i=1}^m$, respectively. If the two motions are not related by a Euclidean transformation and $\boldsymbol{K}_i, \tilde{\boldsymbol{K}}_i \in \mathcal{G}$, they are said to be *critical* with respect to $\mathcal{G}$.

A motion is critical if there exists an alternative projective motion satisfying the auto-calibration constraints. Without any additional assumptions, it is not possible to tell which motion is the true one. One natural additional constraint is that the reconstructed $3D$ structure should lie in front of all cameras. In many (but by no means all) cases this reduces the ambiguity, but it depends on which 3D points are observed.

According to (4) the image of the absolute conic Ω_∞ is,

$$\omega_i^* \simeq \boldsymbol{P}_i \Omega_\infty^* \boldsymbol{P}_i^T \simeq \boldsymbol{K}_i [\, \boldsymbol{R}_i \,|\, -\boldsymbol{R}_i \mathbf{t}_i \,] \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & 0 \end{bmatrix} [\, \boldsymbol{R}_i \,|\, -\boldsymbol{R}_i \mathbf{t}_i \,]^T \boldsymbol{K}_i^T = \boldsymbol{K}_i \boldsymbol{K}_i^T. \quad (5)$$

Thus, knowing the calibration of a camera is equivalent to knowing its image of Ω_∞. Also, if there is a projectively related motion $(\tilde{\boldsymbol{P}}_i)_{i=1}^m$, then the false image of the true absolute conic is the true image of a "false" absolute conic:

$$\tilde{\omega}_i^* \simeq \tilde{\boldsymbol{P}}_i \Omega_\infty^* \tilde{\boldsymbol{P}}_i^T = \tilde{\boldsymbol{P}}_i T^{-1} T \Omega_\infty^* T^T T^{-T} \tilde{\boldsymbol{P}}_i = \boldsymbol{P}_i \Omega_f^* \boldsymbol{P}_i^T,$$

where $\Omega_f^* = T\Omega_\infty^* T^T$ is some dual, virtual quadric of rank 3. This observation allows us to eliminate the "false" motion $(\tilde{\boldsymbol{P}}_i)_{i=1}^m$ from the problem and work only with the true Euclidean motion, but with a false absolute dual quadric Ω_f^*.

Problem 3.2. *Let $\mathcal{G} \subset \mathcal{K}$. Then, given the true motion $(\boldsymbol{P}_i)_{i=1}^m$, where $\boldsymbol{P}_i = \boldsymbol{K}_i[\, \boldsymbol{R}_i \,|\, -\boldsymbol{R}_i \mathbf{t}_i \,]$ and $\boldsymbol{K}_i \in \mathcal{G}$, is there any other proper, virtual conic Ω_f^*, different from Ω_∞^*, such that $\boldsymbol{P}_i \Omega_f^* \boldsymbol{P}_i^T \simeq \tilde{\boldsymbol{K}}_i \tilde{\boldsymbol{K}}_i^T$, where $\tilde{\boldsymbol{K}}_i \in \mathcal{G}$?*

Given only a 3D projective reconstruction derived from uncalibrated images, the true Ω_∞ is not distinguished in any way from any other proper, virtual planar conic in projective space. In fact, given any such potential conic Ω_f^*, it is easy to find a 'rectifying' projective transformation that converts it to the Euclidean DAC form $\Omega_\infty^* = \mathrm{diag}(1,1,1,0)$ and hence defines a false Euclidean structure. To recover the true structure, we need constraints that single out the true Ω_∞ and Π_∞ from all possible false ones. Thus, ambiguity arises whenever the images of some non-absolute conic satisfy the auto-calibration constraints. We call such conics **potential absolute conics** or **false absolute conics**. They are in one-to-one correspondence with possible false Euclidean structures for the scene.

A natural question is whether the problem is dependent on the actual values of the intrinsic parameters. We will show that this is not the case whenever the set $\mathcal{G}$ is a proper subgroup of $\mathcal{K}$. Fortunately, according to the following easy lemma, most of the relevant auto-calibration constraints are **subgroup conditions**.

Lemma 3.1. *The following constrained camera matrices form proper subgroups of the 3×3 upper triangular matrices $\mathcal{K}$:*

(i) Zero skew, i.e. $s = 0$.

(ii) Unit aspect ratio, i.e. $\gamma = 1$.

(iii) Vanishing principal point, i.e. $(u_0, v_0) = (0, 0)$.

(iv) Unit focal length, i.e. $f = 1$.

(v) Combinations of the above conditions.

Independence of the values of the intrinsic camera parameters is shown as follows:

Lemma 3.2. *Let* $G_i \in \mathcal{G}$ *for* $i = 1, ..., m$, *where* $\mathcal{G}$ *is a proper subgroup of* $\mathcal{K}$. *Then, the motion* $(\boldsymbol{P}_i)_{i=1}^m$ *is critical w.r.t.* $\mathcal{G}$ *if and only if the motion* $(G_i\boldsymbol{P}_i)_{i=1}^m$ *is critical w.r.t.* $\mathcal{G}$.

Proof. If $(\boldsymbol{P}_i)_{i=1}^m$ is critical with the alternative motion $(\tilde{\boldsymbol{P}}_i)_{i=1}^m$ and calibrations $\boldsymbol{K}_i, \tilde{\boldsymbol{K}}_i \in \mathcal{G}$, then clearly $(G_i\boldsymbol{P}_i)_{i=1}^m$ and $(G_i\tilde{\boldsymbol{P}}_i)_{i=1}^m$ are also critical, because $G_i\boldsymbol{K}_i, G_i\tilde{\boldsymbol{K}}_i \in \mathcal{G}$ by the closure of $\mathcal{G}$ under multiplication. The converse also holds with G_i^{-1}, by the closure of $\mathcal{G}$ under inversion.

■

Camera matrices with prescribed parameters do not in general form a subgroup of $\mathcal{K}$, but it suffices for them to be of the more general form $\boldsymbol{K}_0\boldsymbol{K}$ where $\boldsymbol{K}_0$ is a known matrix and $\boldsymbol{K}$ belongs to a proper subgroup of $\mathcal{K}$. For example, the set of all camera matrices with known focal length f has the form

$$\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{bmatrix}.$$

The invariance with respect to calibration parameters simplifies things, especially if one chooses $G_i = \boldsymbol{K}_i^{-1}$ for $i = 1, ..., m$. With this in mind, we restrict our attention to proper subgroups of $\mathcal{K}$ and formulate the problem as follows.

Problem 3.3. *Let* $\mathcal{G} \subset \mathcal{K}$ *be a proper subgroup. Then, given the true motion* $(\boldsymbol{P}_i)_{i=1}^m$ *for calibrated cameras, where* $\boldsymbol{P}_i = [\,\boldsymbol{R}_i \,|\, -\boldsymbol{R}_i\mathbf{t}_i\,]$, *is there any other false absolute conic* Ω_f^*, *different from* Ω_∞^*, *such that*

$$[\,\boldsymbol{R}_i \,|\, -\boldsymbol{R}_i\mathbf{t}_i\,]\Omega_f^*[\,\boldsymbol{R}_i \,|\, -\boldsymbol{R}_i\mathbf{t}_i\,]^T \simeq \tilde{\boldsymbol{K}}_i\tilde{\boldsymbol{K}}_i^T,$$

where $\tilde{\boldsymbol{K}}_i \in \mathcal{G}$?

4. Approach

We want to explicitly characterize the critical motions (relative camera placements) for which particular auto-calibration constraints are insufficient to uniquely determine Euclidean 3D structure. We assume that projective structure is available. Alternative Euclidean structures correspond one-to-one with possible locations for a potential absolute conic in $\mathbb{P}^3$. Initially, any proper virtual projective plane conic is potentially absolute, so we look for such conics Ω^* whose images also satisfy the given auto-calibration constraints. Ambiguity arises *if and only if* more than one such conic exists. We work with the true camera motion in a Euclidean frame where the true absolute conic Ω_∞ has its standard coordinates.

Several general invariance properties help to simplify the problem:
Calibration invariance: As shown in the previous section, if the auto-calibration constraints are subgroup conditions, the specific parameter values are irrelevant. Hence, for the purpose of deriving critical motions, we are free to assume that the cameras are in fact secretly calibrated, $\boldsymbol{K}_i = \boldsymbol{I}$, even though we do not assume that we *know* this. (All that we actually know is $\boldsymbol{K}_i \in \mathcal{G}$, which does not allow some image conics $\omega_i^* \neq \boldsymbol{I}$ to be excluded outright).
Rotation invariance: For *known-calibrated* cameras, $\boldsymbol{K}_i$ can be set to identity, and thus the image $\omega_i^* = \boldsymbol{I}$ of any false AC must be identical to the image of the true one. Since

$$\boldsymbol{P}_i \Omega_f^* \boldsymbol{P}_i^T \simeq \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{R}\boldsymbol{P}_i \Omega_f^* \boldsymbol{P}_i^T \boldsymbol{R}^T \simeq \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I},$$

hold for *any* rotation $\boldsymbol{R}$, the image ω_i^* is invariant to camera rotations. Hence, *criticality depends only on the camera centres, not on their orientations.* More generally, any camera rotation that leaves the auto-calibration constraints intact is irrelevant. For example, arbitrary rotations about the optical axis and $180°$ flips about any axis in the optical plane are irrelevant if (a, s) is either $(1, 0)$ or unconstrained, and (u_0, v_0) is either $(0, 0)$ or unconstrained.
Translation invariance: For true or false absolute conics on the plane at infinity, translations are irrelevant so criticality depends only on camera orientation.

In essence, Euclidean structure recovery in projective space is a matter of parameterizing all of the possible proper virtual plane conics, then using the auto-calibration constraints on their images to algebraically eliminate parameters until only the unique true absolute conic remains. More abstractly, if $\boldsymbol{C}$ parameterizes the possible conics and $\mathbf{X}$ the camera geometries, the constraints cut out some algebraic variety in $(\boldsymbol{C}, \mathbf{X})$

space. A constraint set is useful for Euclidean structure from motion recovery only if this variety generically intersects the subspaces $\mathbf{X} = \mathbf{X}_0$ in one (or at most a few) points $(\boldsymbol{C}, \mathbf{X}_0)$, as each such intersection represents an alternative Euclidean structure for the reconstruction from that camera geometry. A set of camera poses $\mathbf{X}$ is **critical** for the constraints if it has exceptionally (e.g. infinitely) many intersections.

Potential absolute conics can be represented in several ways. The following parameterizations have all proven relatively tractable:
(i) Choose a Euclidean frame in which Ω_f^* is diagonal, and express all camera poses relative this frame [36, 37]. This is symmetrical with respect to all the images and usually gives the simplest equations. However, in order to find explicit inter-image critical motions, one must revert to camera-based coordinates which is sometimes delicate. The cases of a finite false absolute conic and a false conic on the plane at infinity must also be treated separately, e.g. $\Omega_f^* = \text{diag}(d_1, d_2, d_3, d_4)$ with either d_3 or d_4 zero.
(ii) Work in the first camera frame, encoding Ω_f^* by its first image ω_1^* and supporting plane $(\mathbf{n}^T, 1)$. Subsequent images $\omega_i^* \simeq \boldsymbol{H}_i\, \omega_1^*\, \boldsymbol{H}_i^T$ are given by the inter-image homographies $\boldsymbol{H}_i = \boldsymbol{R}_i + \mathbf{t}_i\, \mathbf{n}^T$ where $(\boldsymbol{R}_i, \mathbf{t}_i)$ is the i^{th} camera pose. The output is in the first camera frame and remains well-defined even if the conic tends to infinity, but the algebra required is significantly heavier.
(iii) Parameterize Ω_f^* implicitly by two images ω_1^*, ω_2^* subject to the Kruppa constraints. In the two-image case this approach is both relatively simple and rigorous — two proper virtual dual image conics satisfy the Kruppa constraints if and only if they define a (pair of) corresponding 3D potential absolute conics — but it does not extend so easily to multiple images.

The derivations below are mainly based on method (i) .

5. Critical Motions

In this section, the varieties of critical motions are derived. In most situations, the problem is solved in two separate cases. One is when there are potential absolute conics on the plane at infinity, Π_∞, and the other one is conics outside Π_∞. If the potential conics are all on Π_∞, it is still possible to recover Π_∞ and thereby obtain an affine reconstruction. Otherwise, the recovery of affine structure is ambiguous, and we say that the motion is **critical with respect to affine reconstruction**.

The following constraints on the camera calibration are considered:

(i) known intrinsic parameters,

(ii) unknown focal lengths, but the other intrinsic parameters known,

(iii) known skew and aspect ratio.

These constraints form a natural hierarchy and they are perhaps the most interesting ones from a practical point of view. In Section 3, it was shown in that it is sufficient to study the normalized versions of the auto-calibration constraints, since critical motions are independent of the specific values of the intrinsic parameters. That is, when some of the intrinsic parameters are *known*, e.g. the principal point is $(10, 20)$, we may equivalently analyze the case of principal point set to $(0, 0)$. The corresponding camera matrices give rise to subgroup conditions according to Lemma 3.1.

5.1. Known intrinsic parameters

We start with fully calibrated perspective cameras. The results may not come as a surprise, but it is important to know that there are no other possible degenerate configurations.

Proposition 5.1. *Given projective structure and calibrated perspective cameras at $m \geq 3$ distinct finite camera centres, Euclidean structure can always be recovered uniquely. With $m = 2$ distinct camera centres, there is always exactly a twofold ambiguity.*

Proof. Assuming that the cameras have $\boldsymbol{K} = \boldsymbol{I}$ does not change the critical motions. The camera orientations are irrelevant because any false absolute conic must have the same (rotation invariant) images as the true one. Calibrated cameras never admit false absolute conics on Π_∞, as the (known) visual cone of each image conic can intersect Π_∞ in only one conic, which is the true absolute conic. Therefore, consider a finite absolute conic Ω_f^*, with supporting plane outside Π_∞. As all potential absolute conics are proper, virtual and positive semi-definite [34, 37], a Euclidean coordinate system can be chosen such that Ω_f^* has supporting plane $z = 0$, and matrix coordinates

$$\Omega_f^* = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}.$$

Since the cameras are calibrated, $\boldsymbol{K} = \boldsymbol{I}$, and their orientations are irrelevant, $\boldsymbol{R} = \boldsymbol{I}$, the conic projection (4) in each camera becomes

$$[\boldsymbol{I} \,|\, -\mathbf{t}]\, \Omega_f^* \, [\boldsymbol{I} \,|\, -\mathbf{t}]^T \simeq \boldsymbol{I} \quad \Leftrightarrow \quad \begin{bmatrix} d_1 + d_4 t_1^2 & d_4 t_1 t_2 & d_4 t_1 t_3 \\ d_4 t_1 t_2 & d_2 + d_4 t_2^2 & d_4 t_2 t_3 \\ d_4 t_1 t_3 & d_4 t_2 t_3 & d_4 t_3^2 \end{bmatrix} \simeq \boldsymbol{I}.$$

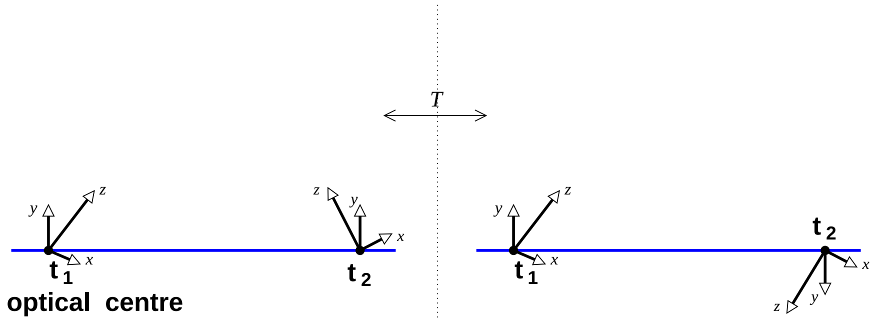

Figure 1. A twisted pair of reconstructions.

As the conic should be proper, both $d_4 \neq 0$ and $t_3 \neq 0$, which gives $t_1 = t_2 = 0$. Thus the only solutions are $\mathbf{t}_\pm = (0, 0, \pm z)$ and $\Omega_f^* \simeq \mathrm{diag}(1, 1, 0, 1/z^2)$ for some $z > 0$. Hence, ambiguity implies that there are at most two camera centres, and the false conic is a circle of imaginary radius $i\,z$, centred in the plane bisecting the two camera centres. ■

In the two-image case, the improper self-inverse projective transformation

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 1/z & 0 \end{bmatrix}$$

interchanges the true Ω_∞^* and the false Ω_f^*, according to

$$T\,\Omega_f^*\,T^T \simeq \Omega_\infty^*$$

and takes the two projection matrices $\boldsymbol{P}_\pm = \boldsymbol{R}_\pm[\boldsymbol{I} \,|\, -\mathbf{t}_\pm]$ to

$$\boldsymbol{P}_- T^{-1} = \boldsymbol{P}_- \quad \text{and} \quad \boldsymbol{P}_+ T^{-1} = -\boldsymbol{R}_+ \left[\begin{smallmatrix} -1 & & \\ & -1 & \\ & & 1 \end{smallmatrix} \,\middle|\, -\mathbf{t}_+ \right].$$

While the first camera remains fixed, the other has rotated 180° about the axis joining the two centres. This twofold ambiguity corresponds exactly to the well-known **twisted pair** duality [23, 18, 27]. The geometry of the duality is illustrated in Figure 1.

The 'twist' T represents a very strong projective deformation that cuts the scene in half, moving the plane between the cameras to infinity, see Figure 2. By considering twisted vs. non-twisted optical ray intersections, one can also show that it reverses the relative signs of the depths, so for one of the solutions the structure will appear to be behind one

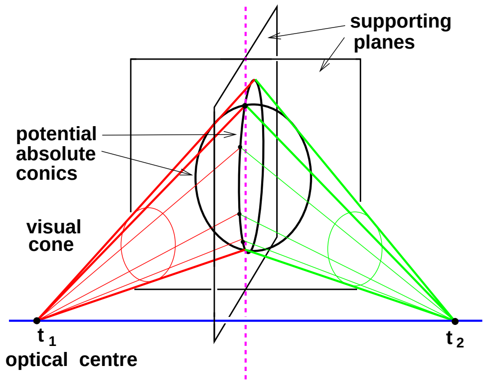

Figure 2. Intersecting the visual cones of two image conics satisfying the Kruppa constraints generates a pair of 3D conics, corresponding to the two solutions of the twisted pair duality.

camera, cf. [17]. To conclude, Proposition 5.1 states that *any* two-view geometry has a 'twisted pair' projective involution symmetry and *any* camera configuration with three or more camera centres has a unique projective-to-Euclidean upgrade.

5.2. Unknown focal lengths

In the case of two images and internally calibrated cameras modulo unknown focal lengths, it is in general possible to recover Euclidean structure. Since we know that the solutions always occur in twisted pairs (which can be disambiguated using the positive depth constraint), it is more relevant to characterize the motions for which there are solutions other than the twisted pair duality. Therefore, the two-camera case will be dealt with separately, after having derived the critical motions for arbitrary many images.

5.2.1. *Many images*

If all intrinsic parameters are known except for the focal lengths, the camera matrix can be assumed to be $\boldsymbol{K} = \mathrm{diag}(f, f, 1)$ which in turn implies that the image of a potential absolute conic satisfies

$$\omega^* = \boldsymbol{K}\boldsymbol{K}^T \simeq \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{for some } \lambda > 0. \tag{6}$$

We start with potential absolute conics on Π_∞.

Potential absolute conics on Π_∞

Let $\boldsymbol{C_f}$ denote a 3×3 matrix corresponding to a false absolute conic (in locus form) on the plane at infinity. Since $\boldsymbol{C_f}$ is not the true one, $\boldsymbol{C_f} \not\simeq \boldsymbol{I}$. The image of $\boldsymbol{C_f}$ is according to (4)

$$\omega \simeq \boldsymbol{R}\boldsymbol{C_f}\boldsymbol{R}^T. \tag{7}$$

Notice that criticality is independent of translation of the camera.

Two cameras are said to have the same **viewing direction** if their optical axes are parallel or anti-parallel.

Proposition 5.2. *Given Π_∞ and known skew, aspect ratio and principal point, then a motion is critical if and only if there is only one viewing direction.*

Proof. Choose coordinates in which camera 1 has orientation $\boldsymbol{R}_1 = \boldsymbol{I}$. Suppose a motion is critical. According to (6) and (7), this implies that $\boldsymbol{C_f} \simeq \mathrm{diag}(1,1,1+\mu) = \boldsymbol{I} + \mu\mathbf{e}_3\mathbf{e}_3^T$ for some $\mu > -1$. For camera 2, let $\boldsymbol{R}_2 = [\,\mathbf{r}_1\,\mathbf{r}_2\,\mathbf{r}_3]$ and apply (7),

$$\boldsymbol{I} + \mu\mathbf{r}_3\mathbf{r}_3^T \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}, \quad \text{for some } \nu > 0.$$

This implies that $\mathbf{r}_3 = \pm\mathbf{e}_3$, and in turn, $\boldsymbol{R}_2 = \left(\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \pm 1 \end{smallmatrix}\right)$ which is equivalent to a fixed viewing direction of the camera. Conversely, suppose the viewing direction is fixed, which means that $\boldsymbol{R}_i = \left(\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \pm 1 \end{smallmatrix}\right)$ for $i > 1$. Then, it is not possible to disambiguate between any of the potential absolute conics in the pencil $\boldsymbol{C_f}(\mu) \simeq \boldsymbol{I} + \mu\mathbf{e}_3\mathbf{e}_3^T$, since $\boldsymbol{R}_i\boldsymbol{C_f}\boldsymbol{R}_i^T = \boldsymbol{C_f}$. ■

Potential absolute conics outside Π_∞

Assume we have a critical motion $(\boldsymbol{R}_i, \mathbf{t}_i)_{i=1}^m$ with the false dual absolute conic Ω_f^*. If the supporting plane for Ω_f^* is Π_∞, the critical motion is described by Proposition 5.2, so assume that Ω_f^* is outside Π_∞. As in the proof of Proposition 5.1, one can assume without loss generality that a Euclidean coordinate system has been chosen such that Ω_f^* has supporting plane $z = 0$, and matrix coordinates

$$\Omega_f^* = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}.$$

The image of Ω_f^* is according to (4),

$$\boldsymbol{P}_i\Omega_f^*\boldsymbol{P}_i^T \simeq \omega_i^* \Leftrightarrow \boldsymbol{R}_i\boldsymbol{C_f}\boldsymbol{R}_i^T \simeq \omega_i^*, \text{ where } \boldsymbol{C_f} = \left[\begin{smallmatrix} d_1 & & \\ & d_2 & \\ & & 0 \end{smallmatrix}\right] + d_4\mathbf{t}_i\mathbf{t}_i^T. \tag{8}$$

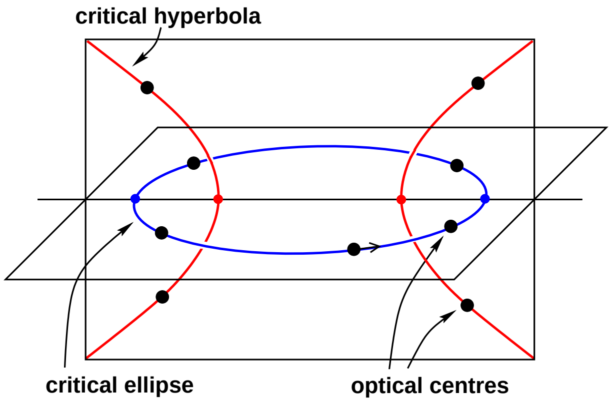

Figure 3. Two orthogonal planes, where one plane contains an ellipse and the other contains a hyperbola.

A necessary condition for degeneracy is that $\boldsymbol{R}_i$ should diagonalize $\boldsymbol{C}_f$ to the form (6), i.e. the matrix $\boldsymbol{C}_f$ must have two equal eigenvalues. As it is always possible to find an orthogonal matrix that diagonalizes a real, symmetric matrix [5], all we need to do is to find out precisely when $\boldsymbol{C}_f$ has two equal eigenvalues. Lemma A.1 in the Appendix characterizes matrices of this form.

Applying the lemma to $\boldsymbol{C}_f$ in (8), with $\sigma_1 = d_1$, $\sigma_2 = d_2$, $\sigma_3 = 0$ and $\rho = d_4$, results in the following cases:

(i) If $d_1 \neq d_2$, then

a. $t_1 = 0$ and $t_2^2 d_1 d_4 + t_3^2 (d_1 - d_2) d_4 = (d_1 - d_2) d_1$, *or*

b. $t_2 = 0$ and $t_1^2 d_2 d_4 + t_3^2 (d_2 - d_1) d_4 = (d_2 - d_1) d_2$.

These equations describe a motion on two planar conics for which the supporting planes are orthogonal. On the first plane, the conic is an ellipse, while on the other the conic is a hyperbola (depending on whether $d_1 > d_2$ or vice versa), see Figure 3.

(ii) If $d_1 = d_2$, then $t_1 = t_2 = 0$ and t_3 arbitrary.

Notice that the second alternative in case (ii) of Lemma A.1 does not occur, since it implies $\mathbf{t}^T \mathbf{e}_3 = 0$, making $\boldsymbol{C}_f$ rank-deficient. Also, case (iii) is impossible, since $\sigma_3 = 0 = \sigma_1 = \sigma_2$.

It remains to find the rotations that diagonalize $\boldsymbol{C}_f$. Since rotations around the optical axis are irrelevant, only the direction of the optical axis is significant. Suppose the optical axis is parameterized by the camera centre $\mathbf{t}$ and a direction $\mathbf{d}$, i.e. $\{\mathbf{t} + \lambda \mathbf{d} | \lambda \in \mathbb{R}\}$. Any point on the axis projects to the principal point,

$$[\boldsymbol{R} \,|\, -\boldsymbol{R}\mathbf{t}] \begin{bmatrix} \mathbf{t} + \lambda \mathbf{d} \\ 1 \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \mathbf{d} \simeq \boldsymbol{R}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

The direction $\mathbf{d}$ should equal the third row of $\boldsymbol{R}$, which corresponds to the eigenvector of the single eigenvalue of $\boldsymbol{C_f}$. Regarding the proof of Lemma A.1, it is not hard to see that the eigenvectors are $\mathbf{v} \simeq (0, t_2\, d_1, t_3\, (d_1 - d_2)\,)$ and $\mathbf{v} \simeq (t_1\, d_2, 0, t_3\, (d_1 - d_2)\,)$ in the two subcases in (i) above. Geometrically, this means that the optical axis must be tangent to the conic at each position, as illustrated in Figure 4(b). Similarly in (ii), it is easy to derive that $\mathbf{v} = (0, 0, \pm 1)$, which means that the optical axis should be tangent to the translation direction, cf. Figure 4(c). An exceptional case is when $\boldsymbol{C_f}$ has a triple eigenvalue, because then any rotation is possible. However, according to Proposition 5.1, it occurs only for twisted pairs. To summarize, we have proven the following.

Proposition 5.3. *Given known intrinsic parameters except for focal lengths, a motion is critical w.r.t. affine reconstruction if and only if the motion consists of (i) rotations with at most two distinct centres (twisted pair ambiguity), or (ii) motion on two conics[1] (one ellipse and one hyperbola) whose supporting planes are orthogonal and where the optical axis is tangent to the conic at each position, or (iii) translation along the optical axis, with arbitrary rotations around the optical axis.*

The motions are illustrated in Figure 4. In case (i) and (ii), the ambiguity of the reconstruction is twofold, as there is only one false absolute conic, whereas in case (iii) there is a one-parameter family of potential planes at infinity (all planes z=constant). Case (iii) can be seen as a special case of the critical motion in Proposition 5.2, which also has a single viewing direction, but arbitrary translations.

5.2.2. *Two images*

For two cameras, projective geometry is encapsulated in the 7 degrees of freedom in the fundamental matrix, and Euclidean geometry in the 5 degrees of freedom in the essential matrix. Hence, from two projective images we might hope to estimate Euclidean structure plus two additional calibration parameters. Hartley [10, 11] gave a method for the case where the only unknown calibration parameters are the focal lengths of the two cameras. This was later elaborated by Newsam et. al.

[1] The actual critical motion is the conics minus the two points where the ellipse intersects the plane $z = 0$, since the image ω^* is non-proper at these points.

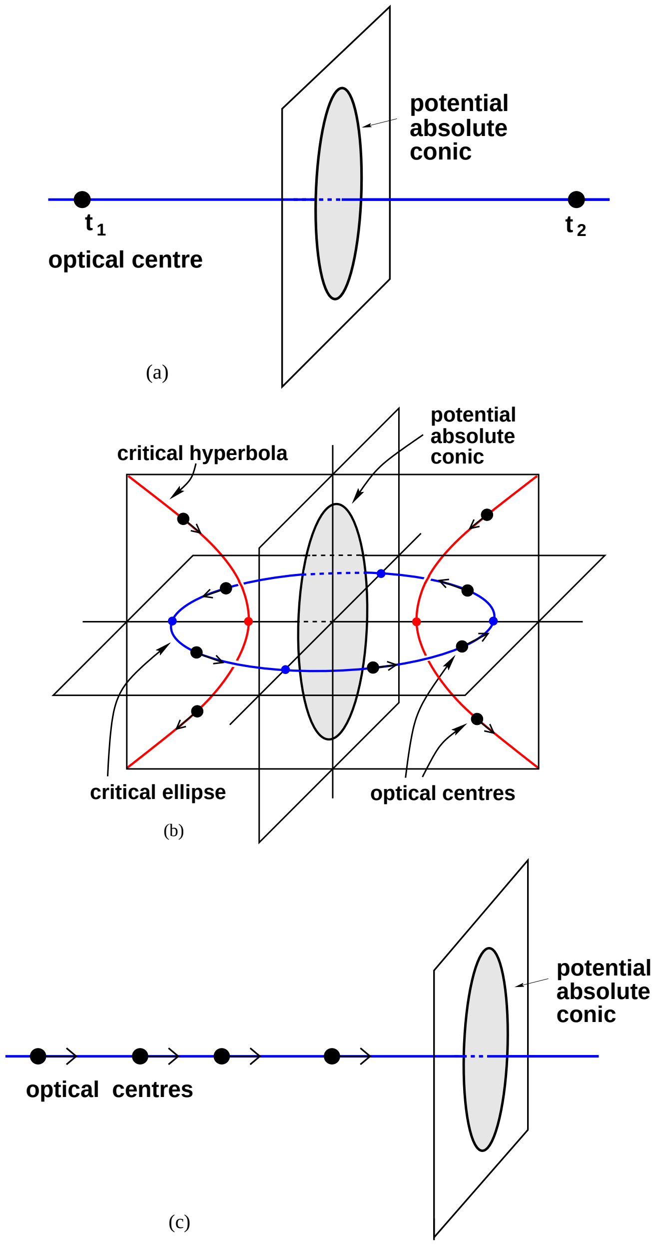

Figure 4. Critical motions for unknown focal lengths: (a) A motion with two fixed centres. (b) A planar motion on an ellipse and a hyperbola. (c) Translation along the optical axis. See Proposition 5.3.

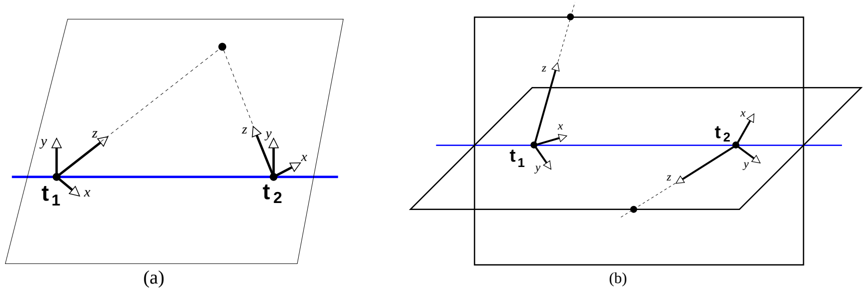

Figure 5. Critical configurations for two cameras: (a) Intersecting optical axes. (b) Orthogonal optical axis planes. See Proposition 5.4.

[28], Zeller and Faugeras [41] and Bougnoux [4]. All of these methods are Kruppa-based. We will derive the critical motions for this case based on the results of the previous sections.

Proposition 5.4. *Given zero skew, unit aspect ratio, principal point at the origin, but unknown focal lengths for two cameras, then a motion (in addition to twisted pair) is critical if and only if (i) the optical axes of the two cameras intersect or (ii) the plane containing the optical axis of camera 1 and camera centre 2, is orthogonal to the plane containing optical axis of camera 2 and camera centre 1.*

Proof. Cf. [28]. Suppose a motion is critical. Regarding Proposition 5.2, we see that if there is only one viewing direction, the optical axes are parallel and intersect at infinity, leading to (i) above. Examining the three possibilities in Proposition 5.3, we see that the first one is the twisted pair solution. The second one, either both cameras lie on the same conic (and hence their axes are coplanar and intersect) or one lies on the hyperbola, the other on the ellipse (in which case their optical axes lie in orthogonal planes) leading to (ii). Conversely, given any two cameras with intersecting or orthogonal-plane optical axes, it is possible to fit (a one-parameter family of) conics through the camera centres, tangential to the optical axes. ∎

The two critical camera configurations are shown in Figure 5.

5.3. Known skew and aspect ratio

Consider the image $\omega^* = \boldsymbol{K}\boldsymbol{K}^T$ of Ω^*_∞. Inserting the parameterization of $\boldsymbol{K}$ in (2) into its dual ω, it turns out that $\omega_{12} = -\frac{s}{f^2\gamma}$. Since f and γ never vanish, requiring that the skew vanishes is equivalent to $\omega_{12} = 0$. The constraint can also be expressed in envelope form using $\omega^* \simeq \omega^{-1}$,

$$\omega_{12} = 0 \;\text{ or dually }\; \omega^*_{12}\omega^*_{33} - \omega^*_{13}\omega^*_{23} = 0. \tag{9}$$

If in addition to zero skew, unit aspect ratio is required in $\boldsymbol{K}$, it is equivalent to $\omega_{11} = \omega_{22}$. This follows from the fact that $\omega_{11} = \frac{1}{f^2}$ and $\omega_{22} = \frac{1}{f^2\gamma}$. The constraint can also be transfered to ω^*,

$$\omega_{11} - \omega_{22} = 0 \text{ or dually } \omega^*_{33}(\omega^*_{22} - \omega^*_{11}) + \omega^{*2}_{13} - \omega^{*2}_{23} = 0. \qquad (10)$$

Analyzing the above constraints on ω in locus form, results in the following proposition when the plane at infinity Π_∞ is known.

Proposition 5.5. *Given Π_∞, a motion is critical with respect to zero skew and unit aspect ratio if and only if there are at most two viewing directions.*

Proof. For each image we have the two auto-calibration constraints (9), (10) with ω given by (7). Choose 3D coordinates in which the first camera has orientation $\boldsymbol{R}_1 = \boldsymbol{I}$. The image 1 constraints become simply $\boldsymbol{C}_{11} - \boldsymbol{C}_{22} = \boldsymbol{C}_{12} = 0$, so we can parameterize $\boldsymbol{C}_f$ with $\boldsymbol{C}_{11}$, $\boldsymbol{C}_{12}$ and $\boldsymbol{C}_{13}$. Given a subsequent image 2, represent its orientation $\boldsymbol{R}_2$ by a quaternion $\mathbf{q} = (q_0, q_1, q_2, q_3)$, evaluate its two auto-calibration constraints, and eliminate $\boldsymbol{C}_{11}$ between them to give:

$$(q_0^2 + q_3^2)(q_1^2 + q_2^2)\,((q_0 q_1 + q_2 q_3)\boldsymbol{C}_{13} + (q_0 q_2 - q_1 q_3)\boldsymbol{C}_{23}) = 0$$

One of the 3 factors must vanish. If the first vanishes the motion is an optical axis rotation, $q_1^2 + q_2^2 = 0$. If the second vanishes it is a $180°$ flip about an axis orthogonal to the optical one, $q_0^2 + q_3^2 = 0$. In both cases the viewing direction remains unchanged and no additional constraint is enforced on $\boldsymbol{C}_f$. Finally, if the third factor vanishes, solving for $\boldsymbol{C}_f$ in terms of $\mathbf{q}$ gives a linear family of solutions of the form

$$\boldsymbol{C}_f \simeq \alpha \boldsymbol{I} + \beta\,(\mathbf{o}_1\,\mathbf{o}_2^T + \mathbf{o}_2\,\mathbf{o}_1^T) \qquad (11)$$

where $\mathbf{o}_1 = (0,0,1)^T$ and $\mathbf{o}_2 =$ (the third row of $\boldsymbol{R}_2(\mathbf{q})$) are the two viewing directions and (α, β) are arbitrary parameters. Conversely, given any potential AC $\boldsymbol{C}_f \not\simeq \boldsymbol{I}$, there is always exactly one pair of real viewing directions $\mathbf{o}_1, \mathbf{o}_2$ that make $\boldsymbol{C}_f$ critical under (11). The linear family $\alpha' \boldsymbol{I} + \beta' \boldsymbol{C}_f$ contains three rank 2 members, one for each eigenvalue λ of $\boldsymbol{C}_f$ (with $\beta'/\alpha' = -\lambda$). Explicit calculation shows that each rank 2 member can be decomposed uniquely (up to sign) into a pair of viewing direction vectors $\mathbf{o}_1, \mathbf{o}_2$ supporting (11), but only the pair corresponding to the middle eigenvalue is real. (Coincident eigenvalues correspond to coincident viewing directions and can be ignored). Hence, no potential AC $\boldsymbol{C}_f$ can be critical for three or more real directions simultaneously. ■

Table I. Summary of critical motions in auto-calibration.

Auto-calibration constraint	**Critical motions**	**Reconstruction ambiguity**
Known calibration	twisted pair duality	projective
Unknown focal length but otherwise known calibration	(i) optical axis rotation (ii) motion on two planar conics (iii) optical axis translation	affine projective projective
Unknown focal length (two images only)	(i) intersecting optical axes (ii) orthogonal optical axis planes	projective projective
Zero skew and unit aspect ratio	(i) two viewing directions (ii) complicated algebraic variety	affine projective

For potential absolute conics outside Π_∞ things are more complicated. For each image, there are two auto-calibration constraints. So in order to single out the true absolute conic (which has 8 degrees of freedom), at least 4 images are necessary. For a given Ω^*_f the polynomial constraints in (9) and (10) determine a variety in the space of rigid motions. We currently know of no easy geometrical interpretation of this manifold.

It is easy to see that given a critical camera motion, the ambiguity is not resolved by rotation around the camera's optical axis.

5.4. Summary

A summary of the critical motions for auto-calibration under the auto-calibration constraints studied is given in Table I. The reconstruction ambiguity is classified as projective if the plane at infinity cannot be uniquely recovered, and affine if it is possible. As mentioned earlier, the twisted pair duality is not a true critical motion, since the positive-depth constraint can always resolve the ambiguity.

6. Particular Motions

Some critical motions occur frequently in practice. In this section, a selection of them is analyzed in more detail.

6.1. Pure rotation

In the case of a stationary camera performing arbitrary rotations, no 3D reconstruction is possible. There always exist many potential absolute conics outside Π_∞.

However, it is still possible to recover the internal camera calibration, provided there are no potential absolute conics on Π_∞, cf. [37]. Proposition 5.2 and Proposition 5.5, regarding critical motions and potential ACs on Π_∞ tells us when such auto-calibration is possible for a purely rotating camera.

6.2. Pure translation

If a sequence of movements only consists of arbitrary translations and no rotations, all proper, virtual conics on Π_∞ are potential absolute conics. Still, one could hope to recover the plane at infinity correctly, and thus get an affine reconstruction.

Proposition 6.1. *Let $(\mathbf{t}_i)_{i=1}^m$ be a general sequence of translations, where m is sufficiently large. Then, the motion is*

(i) always critical w.r.t. affine reconstruction under the constraints zero skew and unit aspect ratio.

(ii) not critical w.r.t. affine reconstruction under the constraints zero skew, unit aspect ratio and vanishing principal point.

Proof. (i) We need to show that there exists a potential DAC Ω_f^* outside Π_∞, which is valid for all $(\mathbf{t}_i)_{i=1}^m$. Choose a coordinate system such that $\boldsymbol{P}_i = \left[\boldsymbol{I} \mid -\mathbf{t}_i\right]$. Then for instance $\Omega_f^* = \mathrm{diag}(1,1,0,1)$ is a potential DAC (multiply $\boldsymbol{P}_i \Omega_f^* \boldsymbol{P}_i^T$ to get ω^* and check that it fulfills (9) and (10)). (ii) follows directly from Proposition 5.3. ■

Note that translating only along the optical axis in case (ii) above results in a critical motion.

6.3. Parallel axis rotations

Sequences of rotations around parallel axes with arbitrary translations are interesting in several aspects. They occur frequently in practice and are one of the major degeneracies for auto-calibration with constant intrinsic parameters [36, 43]. See Figure 6.

It follows directly from Proposition 5.5 that given zero skew, unit aspect ratio and general rotation angles, the fixed-axis motion is not critical unless it is around the optical axis. If we further add the vanishing principal point constraint, the optical axis remains critical according

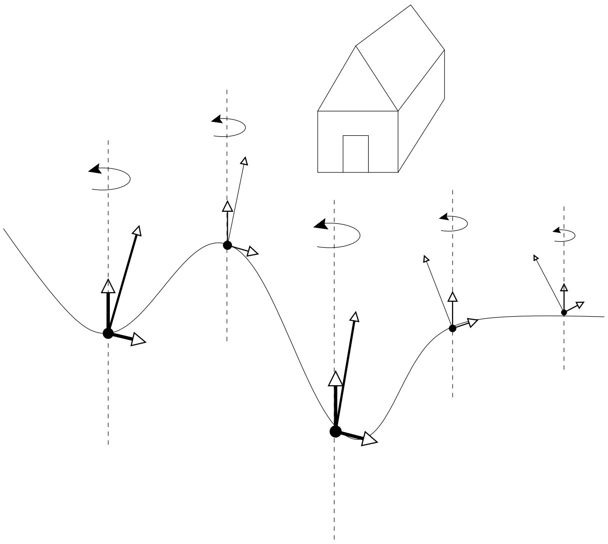

Figure 6. Rotations around the vertical axes with arbitrary translations.

to Proposition 5.2. If we know only that the skew vanishes, we have the following proposition.

Proposition 6.2. *Let* $(\boldsymbol{R}_i, \mathbf{t}_i)_{i=1}^m$ *be a general motion whose rotations are all about parallel axes, where* $\boldsymbol{R}_1 = \boldsymbol{I}$ *and* m *is sufficiently large. Given* Π_∞*, the motion is critical w.r.t. zero skew if and only if the rotation is around one of the following axes:*

(i) $(0, *, *)$ *or* $(*, 0, *)$,

(ii) $(1, 1, 0)$ *or* $(1, -1, 0)$,

where each $*$ *denotes an arbitrary real number.*

Proof. Let $\boldsymbol{C_f}$ denote a false AC on Π_∞. The zero skew constraint in (9) using the parameterization in (7) gives $\boldsymbol{C}_{12} = 0$ for camera 1. An arbitrary rotation around a fixed axis (q_1, q_2, q_3) can be parameterized by $\lambda(q_1, q_2, q_3)$, $\lambda \in \mathbb{R}$. Inserting this into the zero skew constraint in (9) yields a polynomial in $\mathbb{R}[\lambda]$. Since λ can be arbitrary all coefficients of the polynomial must vanish. The solutions to the system of vanishing coefficients are the ones given above. ■

Some of these critical axes may be resolved by requiring that the camera calibration should be constant. In [37], it is shown that parallel axis rotations under constant intrinsic parameters are always critical and give rise to the following pencil of potential absolute conics:

$$\boldsymbol{C}_f(\mu) = \boldsymbol{I} + \mu[q_1, q_2, q_3][q_1, q_2, q_3]^T. \tag{12}$$

Combining constant intrinsic parameters, and some a priori known values of the intrinsic parameters, some of the critical axes are still critical.

Corollary 6.1. *Let $(\boldsymbol{R}_i, \mathbf{t}_i)_{i=1}^m$ be a general motion with parallel axis rotations, where $\boldsymbol{R}_1 = \boldsymbol{I}$ and m is sufficiently large. Given Π_∞, and constant intrinsic parameters, the following axes are the only ones still critical:*

(i) $(0, *, *)$ *and* $(*, 0, *)$ *w.r.t. zero skew,*

(ii) $(0, 0, 1)$ *w.r.t. zero skew and unit aspect ratio,*

(iii) $(0, 0, 1)$ *w.r.t. an internally calibrated camera except for focal length.*

Proof. (i) Using the potential ACs in (12) in the proof of Proposition 6.2, one finds that the only critical axes remaining under the zero skew constraint are $(0, *, *)$ and $(*, 0, *)$. (ii) and (iii) are proved analogously. ∎

7. Experiments

In practice, a motion is never exactly degenerate due to measurement noise and modeling discrepancies. However, if the motion is close to a critical manifold it is likely that the reconstructed parameters will be inaccurately estimated. To illustrate the typical effects of critical motions, we have included some simple synthetic experiments for case of two cameras with unknown focal lengths but other intrinsic parameters known. We focus on the question of how far from critical the two cameras must be to give reasonable estimates of focal length and 3D Euclidean structure [20]. The experimental setup is as follows: two unit focal length perspective cameras view 25 points distributed uniformly within the unit sphere. The camera centres are placed at $(-2, -2, 0)$ and $(2, -2, 0)$ and their optical axes intersect at the origin, similar to the setup in Figure 5(a). Independent Gaussian noise of 1 pixel standard deviation is added to each image point in the 512×512 images.

In the experiment, the elevation angles are varied, upwards for the left camera and downwards for the right one, so that their optical axes are skewed and no longer meet. For each pose, the projective structure

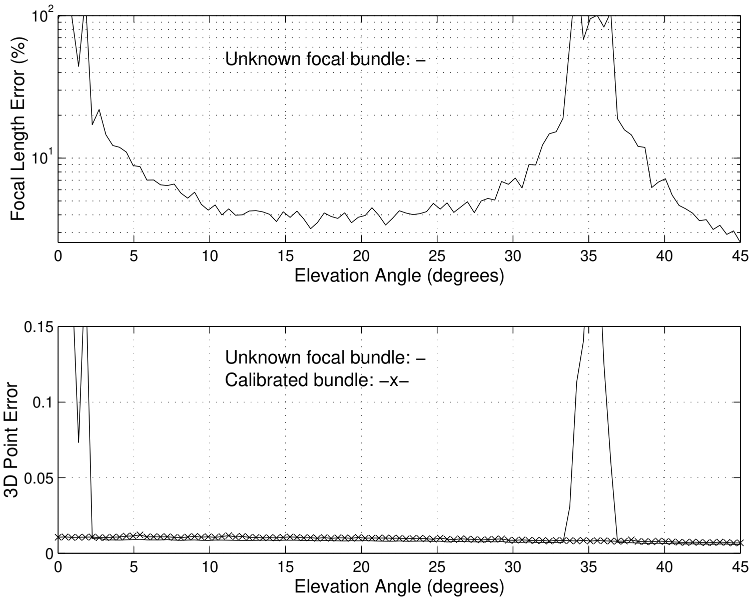

Figure 7. Relative errors vs. camera elevation for two cameras.

and the fundamental matrix are estimated by a projective bundle adjustment that minimises the image distance between the measured and reprojected points [3]. Then, the focal lengths are computed analytically with Bougnoux' method [4]. For comparison, a calibrated bundle adjustment with *known* focal lengths is also applied to the same data. The resulting 3D error is calculated by Euclidean alignment of the true and reconstructed point sets.

Figure 7 shows the resulting root mean square errors over 100 trials as a function of elevation angle. At zero elevation, the two optical axes intersect at the origin. This is a critical configuration according to Proposition 5.4. A second critical configuration occurs when the epipolar planes of the optical axes become orthogonal at around 35° elevation. Both of these criticalities are clearly visible in both graphs. For geometries more than about 5-10° from criticality, the focal lengths can be recovered quite accurately and the resulting Euclidean 3D structure is very similar to the optimal 3D structure obtained with *known* calibration.

8. Conclusion

In this paper, the critical motions in auto-calibration under several auto-calibration constraints have been derived. The various constraints on the intrinsic parameters have been expressed as subgroup conditions on the 3×3 upper triangular camera matrices. With this type of condition, we showed that the critical motions are independent of the specific values of the intrinsic parameters.

It is important to be aware of the critical motions when trying to auto-calibrate a camera. Additional scene or motion constraints may help to resolve the ambiguity, but clearly the best way to avoid degeneracies is to use motions that are "far" from critical. Some synthetic experiments have been performed that give some practical insight to the numerical conditioning of near-critical and critical stereo configurations.

Acknowledgments

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Appendix

Lemma A.1. *Let* $\boldsymbol{A}$ *be a real, symmetric* 3×3 *matrix of the form*

$$\boldsymbol{A} = \sigma_1 \mathbf{e}_1 \mathbf{e}_1^T + \sigma_2 \mathbf{e}_2 \mathbf{e}_2^T + \sigma_3 \mathbf{e}_3 \mathbf{e}_3^T + \rho \mathbf{t}\mathbf{t}^T,$$

where $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$, $\mathbf{t}$ *a non-zero real* 3 *vector and* ρ *a non-zero real scalar. Let* σ_1,σ_2 *and* σ_3 *be given real scalars. Then, necessary and sufficient conditions on* ($\mathbf{t}$,ρ) *for* $\boldsymbol{A}$ *to have two equal eigenvalues can be divided into three cases:*

(i) If $\sigma_1 \neq \sigma_2 \neq \sigma_3$, *then* $\mathbf{t}^T \mathbf{e}_i = 0$ *for at least one* i *(where* $i = 1,2$ *or* 3*). Furthermore,* ρ *can take the values:*

$$\rho = \frac{(\sigma_i - \sigma_j)(\sigma_i - \sigma_k)}{t_j^2(\sigma_i - \sigma_k) + t_k^2(\sigma_i - \sigma_j)},$$

for any i *for which* $\mathbf{t}^T \mathbf{e}_i = 0$ *(where* $j \neq k \neq i$*).*

(ii) If $\sigma_i = \sigma_j \neq \sigma_k$, *then*

a. $\mathbf{t}^T \mathbf{e}_i = \mathbf{t}^T \mathbf{e}_j = 0$ *and* ρ *arbitrary,* or *b.* $\mathbf{t}^T\mathbf{e}_k = 0$ *and* $\rho = \frac{\sigma_k - \sigma_i}{t_i^2 + t_j^2}$.

(iii) *If* $\sigma_1 = \sigma_2 = \sigma_3$, *then* $\mathbf{t}$ *and* ρ *arbitrary.*

Proof. It follows from the Spectral Theorem [5] that if $\boldsymbol{A}$ is real and symmetric with two equal eigenvalues μ, then there is a third eigenvector $\mathbf{v}$ and a scalar ν such that $\boldsymbol{A} = \mu\boldsymbol{I} + \nu\mathbf{v}\mathbf{v}^T$. (The eigenvalue corresponding to $\mathbf{v}$ is $\mu + \nu$). This gives

$$\sigma_1\mathbf{e}_1\mathbf{e}_1^T + \sigma_2\mathbf{e}_2\mathbf{e}_2^T + \sigma_3\mathbf{e}_3\mathbf{e}_3^T + \rho\mathbf{t}\mathbf{t}^T - \nu\mathbf{v}\mathbf{v}^T - \mu\boldsymbol{I} = 0.$$

Multiplying this matrix equation with $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, results in three vector equations,

$$\begin{aligned}(\sigma_1 - \mu)\mathbf{e}_1 + \rho t_1\mathbf{t} - \nu v_1\mathbf{v} = 0\\ (\sigma_2 - \mu)\mathbf{e}_2 + \rho t_2\mathbf{t} - \nu v_2\mathbf{v} = 0\\ (\sigma_3 - \mu)\mathbf{e}_3 + \rho t_3\mathbf{t} - \nu v_3\mathbf{v} = 0.\end{aligned} \tag{13}$$

To prove (i), assume $\sigma_1 \neq \sigma_2 \neq \sigma_3$. The orthogonal bases $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent and cannot all be linear combinations of $\mathbf{t}$ and $\mathbf{v}$, so one of $\sigma_i - \mu$ must vanish, and thereby exactly one. Suppose $\mu = \sigma_1$. Then,

$$\rho t_1\mathbf{t} - \nu v_1\mathbf{v} = 0.$$

If one of the coefficients is non-zero, then $\mathbf{t}$ and $\mathbf{v}$ would be linearly dependent. However, this is impossible because $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent and $\sigma_2 - \mu \neq 0$, $\sigma_3 - \mu \neq 0$. Analogously, $\rho \neq 0$ because otherwise $\mathbf{e}_2$ and $\mathbf{e}_3$ would be linearly dependent according to (13). Therefore $\mathbf{t}^T\mathbf{e}_1 = 0$.

If $\mathbf{t}$ is orthogonal to $\mathbf{e}_1$ and $\mu = \sigma_1$, some easy calculations yield that ρ must be chosen as

$$\rho = \frac{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}{t_2^2(\sigma_1 - \sigma_3) + t_3^2(\sigma_1 - \sigma_2)},$$

which is also sufficient.

When two or three of σ_i $(i = 1, 2, 3)$ are equal, similar arguments can be used to deduce (ii) and (iii).

■

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